Weak Values with Decoherence (Non-Commutative Analysis and Micro-Macro Duality)

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Weak Values with Decoherence [1]

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Abstract

We introduce a weak operator associated with the weak values and give a general framework of quantum operations to the weak operator in parallel with the Kraus representation of the completely positive map for the density operator. The decoherence effect is also investigated in terms of the weak measurement by a shift of a probe wave function of continuous variable.

1 Introduction

The standard measurement, that is, the von Neumann measurement, is not time symmetric. However, we can construct the time symmetric quantum measurement by a post selection [1]. Furthermore, introducing the weak measurement, the *weak value* advocated by Aharonov and his collaborators [2, 3] can be experimentally accessible. This measurement scheme gives us a new interpretation and view for the quantum world [4]. For an observable A, the weak value $\langle A\rangle_w$ is defined as

$$\langle A\rangle_w = \frac{\langle f|U(t_f,t)AU(t,t_i)|i\rangle}{\langle f|U(t_f,t_i)|i\rangle} \in \mathbb{C}, \tag{1}$$

where $|i\rangle$ and $\langle f|$ are pre-selected ket and post-selected bra state vectors, respectively. Here, $U(t_2,t_1)$ is an evolution operator from the time t_1 to t_2. The weak value $\langle A\rangle_w$ actually depends on the pre- and post-selected states $|i\rangle$ and $\langle f|$ but we omit them for notational simplicity in the case that we fix them. Otherwise, we write them explicitly as ${}_f\langle A\rangle_i^w$ instead for $\langle A\rangle_w$. The denominator is assumed to be non-vanishing.

We define a *weak operator* $W(t)$ as

$$W(t) := U(t,t_i)|i\rangle\langle f|U(t_f,t). \tag{2}$$

To facilitate the formal development of the weak value, we introduce the ket state $|\psi(t)\rangle$ and the bra state $\langle\phi(t)|$ as

$$\begin{aligned} |\psi(t)\rangle &= U(t,t_i)|i\rangle \\ \langle\phi(t)| &= \langle f|U(t_f,t), \end{aligned} \tag{3}$$

so that the expression for the weak operator simplifies to

$$W(t) = |\psi(t)\rangle\langle\phi(t)|. \tag{4}$$

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By construction, the two states $|\psi(t)\rangle$ and $\langle\phi(t)|$ satisfy the Schrödinger equations with the same Hamiltonian with the initial and final conditions $|\psi(t_i)\rangle = |i\rangle$ and $\langle\phi(t_f)| = \langle f|$. In a sense, $|\psi(t)\rangle$ evolves forward in time while $\langle\phi(t)|$ evolves backward in time. The time reverse of the weak operator (4) is $W^\dagger = |\phi(t)\rangle\langle\psi(t)|$. Thus, we can say the weak operator is based on the two-state vector formalism [5]. The weak operator gives the weak value of the observable A as

$$\langle A\rangle_w = \frac{\mathrm{Tr}(WA)}{\mathrm{Tr}\,W}, \tag{5}$$

in parallel with the expectation value of the observable A by $\mathrm{Tr}(\rho A)/\mathrm{Tr}\,\rho$ from Born's probabilistic interpretation.

Our aim is to find the most general map for the weak operator W. The result terms out to be of the form $\mathcal{E}(W) = \sum_i E_i W F_i^\dagger$ [6].

2 Quantum Operations for Weak Operators

Let us now define a weak operator as Eq. (4). We discuss a state change by the weak operator and define a map X as

$$X(|\alpha\rangle, |\beta\rangle) := (\mathcal{E}\otimes I)\left(|\alpha\rangle\langle\beta|\right), \tag{6}$$

for an arbitrary $|\alpha\rangle, |\beta\rangle \in \mathcal{H}_s\otimes\mathcal{H}_e$. We consider the following states;

$$\begin{aligned} |\psi(t)\rangle_s &= \sum_k \psi_k|\alpha_k\rangle_s, & |\phi(t)\rangle_s &= \sum_k \phi_k|\beta_k\rangle_s, \\ |\tilde{\psi}(t)\rangle_e &= \sum_k \psi_k^*|\alpha_k\rangle_e, & |\tilde{\phi}(t)\rangle_e &= \sum_k \phi_k^*|\beta_k\rangle_e, \end{aligned} \tag{7}$$

where $\{|\alpha_k\rangle_s\}, \{|\beta_k\rangle_s\}, \{|\alpha_k\rangle_e\}$, and $\{|\beta_k\rangle_e\}$ are complete orthonormal sets of $\mathcal{H}_s$ and $\mathcal{H}_e$. Then, we obtain the following theorem on the state change of the weak operator.

Theorem 1 *Let the quantum operation $\mathcal{E}$ be given. For any weak operator $W = |\psi(t)\rangle_s\langle\phi(t)|$, a change of the weak operator can be written as*

$$\mathcal{E}\left(|\psi(t)\rangle_s\langle\phi(t)|\right) = {}_e\langle\tilde{\psi}(t)|X(|\alpha\rangle, |\beta\rangle)|\tilde{\phi}(t)\rangle_e, \tag{8}$$

where $|\alpha\rangle, |\beta\rangle \in \mathcal{H}_s\otimes\mathcal{H}_e$ are pure states.

This proof is given in the paper [6] and analogous to the density operator.

We take the polar decomposition of the positive operator $X := X(\alpha, \beta)$ to obtain

$$X = \sqrt{\sigma(\alpha)}U, \tag{9}$$

where U is some unitary operator on $\mathcal{H}_s\otimes\mathcal{H}_e$ and $\sigma(\alpha)$ is defined in $\sigma(|\alpha\rangle) := (\mathcal{E}\otimes I)(|\alpha\rangle\langle\alpha|)$. This is because $XX^\dagger = \sqrt{\sigma(\alpha)}UU^\dagger\sqrt{\sigma(\alpha)} = \sigma(\alpha)$. From the positivity of σ, we can rewrite X as

$$X = \sum_m \sqrt{s_m}|\hat{s}_m\rangle\langle\hat{s}_m|U = \sum_m |s_m\rangle\langle t_m|, \tag{10}$$

where $\langle t_m| = \langle \hat{s}_m|U$. Similarly to the Kraus operator [7], we define the two operators, E_i and $F_i^\dagger$, as

$$E_m|\psi(t)\rangle_s := {}_e\langle\tilde{\psi}(t)|s_m\rangle \quad (11)$$

$${}_s\langle\phi(t)|F_m^\dagger := \langle t_m|\tilde{\phi}(t)\rangle_e. \quad (12)$$

Therefore, we obtain the change of the weak operator as

$$\mathcal{E}(W) = \sum_m E_m W F_m^\dagger, \quad (13)$$

using Theorem 1 and linearity. Note that, in general, $\mathcal{E}(W)\mathcal{E}(W^\dagger) \neq \mathcal{E}(\rho)$ although $\rho = WW^\dagger$. When the quantum operation $\mathcal{E}$ is a trace preserving map, we can express the Kraus operators,

$$\begin{aligned} E_m &= {}_e\langle e_m|U|e_i\rangle_e, \\ F_m^\dagger &= {}_e\langle e_f|V|e_m\rangle_e, \end{aligned} \quad (14)$$

for some unitary operators U and V, which act on $\mathcal{H}_s \otimes \mathcal{H}_e$. $|e_i\rangle$ and $|e_f\rangle$ are some basis vectors and $|e_m\rangle$ is a complete set of basis vectors with $\sum_m |e_m\rangle\langle e_m| = 1$ such that $\sum_m E_m^\dagger E_m = 1$ and $\sum_m F_m^\dagger F_m = 1$. We can compute

$$\begin{aligned} \sum_m F_m^\dagger E_m &= \sum_m {}_e\langle e_f|V|e_m\rangle_e \langle e_m|U|e_i\rangle_e \\ &= {}_e\langle e_f|VU|e_i\rangle_e = {}_e\langle e_f|S|e_i\rangle_e, \end{aligned} \quad (15)$$

where $S = VU = U(t_f, t_i)$ is the S-matrix.

3 Weak Measurement with Decoherence

So far we have formally discussed the quantum operations of the weak operators. In this section, we would like to study the effect of environment in the course of the weak measurement [2] and see how the shift of the probe position is affected by the environment. As we shall see, the shift is related to the quantum operation of the weak operator $\mathcal{E}(W)$ (13) which we have investigated in the previous section.

3.1 Weak Measurement—Review

First, we recapitulate the idea of the weak measurement [2, 8]. Consider a target system and a probe defined in the Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_p$. The interaction of the target system and the probe is assumed to be weak and instantaneous,

$$H_{int}(t) = g\delta(t - t_0)(A \otimes P), \quad (16)$$

where an observable A is defined in $\mathcal{H}_s$, while P is the momentum operator of the probe. The time evolution operator becomes $e^{-ig(A\otimes P)}$. Suppose the probe state is initially $\xi(q)$ in the coordinate representation with the probe position q, which is a real-valued function.

For the transition from the pre-selected state $|i\rangle$ to the post-selected state $|f\rangle$, the probe wave function becomes $\langle f|Ve^{-ig(A\otimes P)}U|i\rangle\xi(q)$, which is in the weak coupling case,

$$\begin{aligned}&\langle f|Ve^{-ig(A\otimes P)}U|i\rangle\xi(q)\\&\approx \langle f|VU|i\rangle\xi\left(q-g\frac{\langle f|VAU|i\rangle}{\langle f|VU|i\rangle}\right). \end{aligned} \tag{17}$$

In the previous notation, the argument of the wave function is shifted by $g\langle f|VAU|i\rangle/\langle f|VU|i\rangle = g\langle A\rangle_w$ so that the shift of the expectation value is the real part of the weak value, $g\cdot \mathrm{Re}[\langle A\rangle_w]$. The shift of the momentum distribution can be similarly calculated to give $2g\cdot Var(p)\cdot \mathrm{Im}[\langle A\rangle_w]$, where $Var(p)$ is the variance of the probe momentum before the interaction. Putting together, we can measure the weak value $\langle A\rangle_w$ by observing the shift of the expectation value of the probe both in the coordinate and momentum representations. The shift of the probe position contains the future information up to the post-selected state.

3.2 Weak Measurement and Environment

Let us consider a target system coupled with an environment and a general weak measurement for the compound of the target system and the environment. We assume that there is no interaction between the probe and the environment. This situation is illustrated in Fig. 1. The Hamiltonian for the target system and the environment is given by

$$H = H_0 \otimes I_e + H_1, \tag{18}$$

where H_0 acts on the target system $\mathcal{H}_s$ and the identity operator I_e is for the environment $\mathcal{H}_e$, while H_1 acts on $\mathcal{H}_s\otimes\mathcal{H}_e$. The evolution operators U and V can be expressed by $U = U_0K(t_0,t_i)$ and $V = K(t_f,t_0)V_0$, where U_0 and V_0 are the evolution operators forward in time and backward in time, respectively, by the target Hamiltonian H_0. K's are the evolution operators in the interaction picture,

$$\begin{aligned}K(t,t_i) &= \mathcal{T}e^{-i\int_{t_i}^{t}dtU_0^\dagger H_1 U_0},\\ K(t_f,t) &= \overline{\mathcal{T}}e^{-i\int_{t}^{t_f}dtV_0 H_1 V_0^\dagger},\end{aligned} \tag{19}$$

where $\mathcal{T}$ and $\overline{\mathcal{T}}$ stand for the time-ordering and anti time-ordering products.

Let the initial and final environmental states be $|e_i\rangle$ and $|e_f\rangle$, respectively. The probe state now becomes

$$N\xi\left(q-g\frac{\langle f|\langle e_f|K(t_f,t_0)V_0AU_0K(t_0,t_i)|e_i\rangle|i\rangle}{N}\right), \tag{20}$$

where $N = \langle f|\langle e_f|K(t_f,t_0)V_0U_0K(t_0,t_i)|e_i\rangle|i\rangle$ is the normalization factor. We define the dual quantum operation as

$$\begin{aligned}\mathcal{E}^*(A) &:= \langle e_f|K(t_f,t_0)V_0AU_0K(t_0,t_i)|e_i\rangle\\ &= \sum_m V_0F_m^\dagger AE_mU_0,\end{aligned} \tag{21}$$

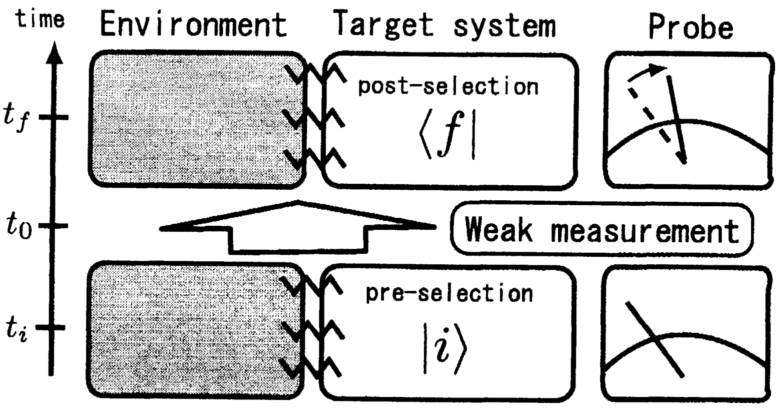

Figure 1: A weak measurement model with the environment. The environment affects the target system as a noise but does not affect the probe. The weak measurement for the target system and the probe brings about the shift of the probe position at t_0. The amount of the shift depends that the environmental state is controllable.

where

$$F_m^\dagger := V_0^\dagger \langle e_f|K(t_f,t_0)|e_m\rangle V_0,$$
$$E_m := U_0\langle e_m|K(t_0,t_i)|e_i\rangle U_0^\dagger \tag{22}$$

are the Kraus operators. Here, we have inserted the completeness relation $\sum_m |e_m\rangle\langle e_m| = 1$ with $|e_m\rangle$ being not necessarily orthogonal. The meaning of the basis $|e_i\rangle$ and $|e_f\rangle$ is now clear as remarked before. Thus, we obtain the wave function of the probe as

$$\xi\left(q - g\frac{\langle f|V_0\mathcal{E}^*(A)|i\rangle}{N}\right)$$
$$= \xi\left(q - g\frac{\mathrm{Tr}[\mathcal{E}(W)A]}{\mathrm{Tr}[\mathcal{E}(W)]}\right) = \xi(q - g\langle A\rangle_{\mathcal{E}(W)}), \tag{23}$$

with $N = \langle f|\mathcal{E}^*(I)|i\rangle$ up to the overall normalization factor. This is the main result of this subsection. The shift of the expectation value of the position operator on the probe is

$$\delta q = g\cdot \mathrm{Re}[\langle A\rangle_{\mathcal{E}(W)}]. \tag{24}$$

From an analogous discussion, we obtain the shift of the expectation value of the momentum operator on the probe as

$$\delta p = 2g\cdot Var(p)\cdot \mathrm{Im}[\langle A\rangle_{\mathcal{E}(W)}]. \tag{25}$$

Thus, we have shown that the weak value given by the probe shift is affected by the environment during the weak measurement.

4 Summary and Discussions

We have introduced the weak operator W (2) to formally describe the weak value advocated by Aharonov and his collaborates. The general framework is given to describe effects of quantum operation $\mathcal{E}(W)$ (13) to the weak operator W in parallel with the Kraus representation of the completely positive map for the density operator ρ. We have shown the effect of the environment during the weak measurement as the shift of the expectation value of the probe observables in both cases of the controllable and uncontrollable environmental states.

Extending our proposed definition of the weak operators, we may consider a superposition of weak operators,

$$W := \sum_{i,f} \alpha_{if} U(t, t_i)|i\rangle\langle f|U(t_f, t), \tag{26}$$

in analogy to the mixed state which is a convex linear combination of pure states. Actually, $\mathcal{E}(W)$ (13) has the form (26). Although this indicates a time-like correlations, the physical implication is not yet clear. This operator may be related to the concept of the multi-time states [9]. In fact, it is shown how the weak value corresponding to the weak operator (26) can be constructed via a protocol by introducing auxiliary states which are space-likely entangled with the target states.

References

[1] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).

[2] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60** (1988) 1351.

[3] Y. Aharonov and D. Rohrlich, *Quantum Paradoxes* (Wiley-VCH, Weibheim, 2005).

[4] Y. Aharonov and E. Y. Gruss, arXiv:quant-ph/0507269.

[5] Y. Aharonov and L. Vaidman, Phys. Rev. A **41** (1990) 11.

[6] Y. Shikano and A. Hosoya, arXiv:0812.4502.

[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[8] R. Jozsa, Phys. Rev. A **76** (2007) 044103.

[9] Y. Aharonov, S. Popescu, J. Tollaksen, and L. Vaidman, arXiv:0712.0320.